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Response of a Blood Vessel Segment to Finite Strain with Small, Sinusoidally Varying Incremental Strains Superimposed upon It

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Of concern in the present investigation is the response of an arterial segment of a finite length in a state of finite strain when a small additional sinusoidally varying strain is superimposed upon it. The situation pertains to the case when one end of the arterial segment is held fixed while the other end is subjected to a given load in the longitudinal direction. The analysis is particularly valid for a typically large artery for which the radial stress is negligibly small compared to the circumferential and longitudinal stresses. © 1985 Academic Press, Inc.

INTRODUCTION

It is now a well-known fact that cardiovascular diseases account for more than 50% of the total number of deaths. One of the most widespread cardiovascular diseases is atherosclerosis, the formation of which is generally believed to be caused by the high concentration of stresses due to hypertension in the vascular walls. Therefore, for a fuller understanding of the pathology and physiology of the cardiovascular system, studies related to the stresses and deformations in the walls of blood vessels are very much essential.

The distribution of stresses in the walls of blood vessels has been studied by many authors. Vaishnav et al. [1, 2] developed a nonlinear theory suitable for the study of vascular mechanics; they [1] put forward three expressions in the form of second, third and fourth degree polynomials for strain energy functions for the wall tissues and also experimentally determined the material constants involved in their analysis. In vitro studies on the deformation of soft biological tissues were made by Hartung [3] and Misra [4]. Further discussions on the mechanical properties of soft biological tissues with particular emphasis on the material behaviour of blood vessel walls can be found in Hartung [5, 6]. Young et al. [7] as an extension of the investigations reported in [1, 2] presented nonlinear con-

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stitutive relations for studying the mechanical behaviour of blood vessel walls. Recently, some researchers, e.g., Misra and Chakrabarty [8, 9] and Misra and Roy Choudhury [10], studied various aspects of vascular biomechanics by employing nonlinear viscoelastic relations for characterizing the mechanical behaviour of wall tissues. The principal object of the present study is to examine the effect of a small, sinusoidally varying incremental strain superimposed on a state of finite strain of the segment of a blood vessel. A theory of small deformation superimposed on a finite deformation has been developed here. The importance of this study lies in the fact that it stimulates the usual physiological deformation of the blood vessel wall wherein small pulsatile variations in strains are superimposed upon the finite deformation caused by mean intravascular pressure. The applicability of the mathematical analysis is illustrated through the numerical computation of a few quantities of physical interest.

Mathematical Analysis

By considering a blood vessel segment having cylindrical geometry, the wall material has been treated here as nonlinear, incompressible, viscoelastic and orthotropic, based upon the experimental observations of Fenn [12], Harkness et al. [13], Carew et al. [14], Patel et al. [15], Tucker et al. [16], Vaishnav et al. [17], Simon et al. [18] and Fung [19].

If (X, Y, Z), (x, y, z) are the cartesian coordinates of a point of the vessel segment in the undeformed and deformed states, respectively, and if (R, Θ, Z), (r, θ, z) represent the same point in the respective configurations in the cylindrical polar coordinate system, one can write

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} R\cos\Theta \\ R\sin\Theta \\ Z \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \\ z \end{bmatrix}. \tag{1}$$

Let the radial, tangential and longitudinal stretch ratios be denoted by λ_r, λ_θ, and λ_z, respectively. Then

$$\lambda_r = \frac{\partial r}{\partial R} \quad \text{and} \quad \lambda_\theta = \frac{r}{R}. \tag{2}$$

The condition of incompressibility yields

$$\lambda_r \lambda_\theta \lambda_z = 1.$$

If E_{ij} is used to denote the components of the Lagrangian strain-tensor and $E_{(i)(j)}$ denote the respective physical components, we can write

$$E_{(i)(j)} = \text{diag}[\tfrac{1}{2}(\lambda_r^2 - 1),\ \tfrac{1}{2}(\lambda_\theta^2 - 1),\ \tfrac{1}{2}(\lambda_z^2 - 1)]. \tag{3}$$

The equation of motion characterizing the state of finite deformation may be put in the form

$$\frac{\partial T^{\alpha\beta}}{\partial \Theta^\alpha} + \Gamma^\alpha_{\alpha\lambda} T^{\lambda\beta} + \Gamma^\beta_{\alpha\lambda} T^{\alpha\lambda} = \rho f^\beta \tag{4}$$

where $T^{\alpha\beta}$ are the stress-components in finite deformation state, ρ is the density of the wall material and f^β are the components of the acceleration vector.

$\Gamma^\alpha_{\beta\gamma}$ are the Christoffel symbols, defined as

$$\Gamma^\alpha_{\beta\gamma} = \tfrac{1}{2} g^{\alpha\lambda}[g_{\lambda\beta,\gamma} + g_{\gamma\lambda,\beta} - g_{\beta\gamma,\lambda}] \tag{5}$$

where $g^{\alpha\beta}$ is the metric tensor corresponding to the deformed configuration.

In the cylindrical co-ordinate system, the only two nonvanishing quantities are

$$\Gamma^1_{22} = r\dot{\lambda}_r$$

and (6)

$$\Gamma^2_{12} = \frac{\dot{\lambda}_r}{r}.$$

In order to examine the effect of small sinusoidally varying strains superimposed on the blood vessel segment in its finite deformation state, let us consider the circumferential and longitudinal strains in the following form:

$$E_{(2)(2)}(t) = [a_0 + a_1 e^{i\omega t}]\, H(t - T)$$

and (7)

$$E_{(3)(3)}(t) = [b_0 + b_1 e^{i\omega t}]\, H(t - T)$$

in which a_0 and b_0 are the constant finite strains and a_1, b_1 are the amplitudes of the superimposed strains, where $a_1 \ll 1$ and $b_1 \ll 1$. The small sinusoidal strain is assumed to be applied suddenly at time T, which is described by the inclusion of the Heaviside step function $H(t - T)$. Substituting these expressions (7) in the stress–strain relations (cf. Vaishnav et al. [17]) and denoting by s_i the physical components of the stress,

$$s_\theta - s_r = \left(\frac{\partial\theta^{(2)}}{\partial\Theta^{(2)}}\right)^2 \left[\int_{-\infty}^{t} \bar{K}^{2222}(t-T)\,\dot{E}_{22}(T)\,dT + \int_{-\infty}^{t} \bar{K}^{2233}(t-T)\,\dot{E}_{33}(T)\,dT \right.$$
$$+ \int_{-\infty}^{t}\int_{-\infty}^{t} \bar{K}^{222222}(t-T_1, t-T_2)\,\dot{E}_{22}(T_1)\,\dot{E}_{22}(T_2)\,dT_1\,dT_2$$
$$+ \int_{-\infty}^{t}\int_{-\infty}^{t} \bar{K}^{222233}(t-T_1, t-T_2)\,\dot{E}_{22}(T_1)\,\dot{E}_{33}(T_2)\,dT_1\,dT_2$$
$$\left. + \int_{-\infty}^{t}\int_{-\infty}^{t} \bar{K}^{223333}(t-T_1, t-T_2)\,\dot{E}_{33}(T_1)\,\dot{E}_{33}(T_2)\,dT_1\,dT_2\right]$$

and

$$s_z - s_r = \left(\frac{\partial\theta^{(3)}}{\partial\Theta^{(3)}}\right)^2 \left[\int_{-\infty}^{t} \bar{K}^{3322}(t-T)\,\dot{E}_{22}(T)\,dT + \int_{-\infty}^{t} \bar{K}^{3333}(t-T)\,\dot{E}_{33}(T)\,dT \right.$$
$$+ \int_{-\infty}^{t}\int_{-\infty}^{t} \bar{K}^{332222}(t-T_1, t-T_2)\,\dot{E}_{22}(T_1)\,\dot{E}_{22}(T_2)\,dT_1\,dT_2$$
$$+ \int_{-\infty}^{t}\int_{-\infty}^{t} \bar{K}^{332233}(t-T_1, t-T_2)\,\dot{E}_{22}(T_1)\,\dot{E}_{33}(T_2)\,dT_1\,dT_2$$
$$\left. + \int_{-\infty}^{t}\int_{-\infty}^{t} \bar{K}^{333333}(t-T_1, t-T_2)\,\dot{E}_{33}(T_1)\,\dot{E}_{33}(T_2)\,dT_1\,dT_2\right]. \quad (8)$$

The stress differences $s_\theta - s_r$ and $s_z - s_r$ can be obtained in terms of $a_0, b_0, a_1, b_1, \omega, T$ and the relaxation functions $\bar{K}$. If these expressions are simplified further by ignoring the terms containing a_1^2, $a_1 b_1$ and b_1^2 (a_1 and b_1 being small), we obtain

$$s_\theta - s_r = \phi_0 \qquad \text{and} \qquad s_z - s_r = \Psi_0. \quad (9)$$

The stress-differences for the superimposed state are given by

$$s'_\theta - s'_r = (\phi_{ac} + \phi_{as})\,a_1 + (\phi_{bc} + \phi_{bs})\,b_1$$

and (10)

$$s'_z - s'_r = (\Psi_{ac} + \Psi_{as})\,a_1 + (\Psi_{bc} + \Psi_{bs})\,b_1$$

in which

$$a_1 = \lambda_\theta \frac{u}{R}, \qquad b_1 = \lambda_z \frac{\partial W}{\partial z},$$

u and w being the physical components of the superimposed displacement in the radial and axial directions. The derived expressions for $\phi_0, \psi_0, \phi_{ac}, \phi_{as}, \phi_{bc}$ and ϕ_{bs} are included in Appendix 1.

In deriving the expressions for the stresses, we have made use of the four-function theory of Vaishnav et al. [17], who remarked that the ten-function theory is applicable for "critical applications" and the four-function theory is adequate for "general use" to describe the nonlinear behaviour of the aortic tissue.

The equations of motion for the superimposed state may be written as (cf. Green and Zerna [11])

$$\frac{\partial T'^{\alpha\beta}}{\partial \Theta^\alpha} + \Gamma^{\alpha}_{\alpha\lambda}\, T'^{\lambda\beta} + \Gamma^{\beta}_{\alpha\lambda}\, T'^{\alpha\lambda} + \Gamma'^{\alpha}_{\alpha\lambda}\, T^{\lambda\beta} + \Gamma'^{\beta}_{\alpha\lambda}\, T^{\alpha\lambda} = \rho f'^{\beta}, \tag{11}$$

$T'^{\alpha\beta}$ and $\Gamma'^{\alpha}_{\alpha\beta}$ being respectively stress components and the Christoffel symbols $\Gamma'^{\alpha}_{\beta\gamma}$ for the new superimposed state are given by

$$\Gamma'^{\alpha}_{\beta\gamma} = \tfrac{1}{2} g^{\alpha\lambda}[g'_{\lambda\beta,\gamma} + g'_{\gamma\lambda,\beta} - g'_{\beta\gamma,\lambda}] + \tfrac{1}{2} g'^{\alpha\lambda}[g_{\lambda\beta,\gamma} + g_{\gamma\lambda,\beta} - g_{\beta\gamma,\lambda}], \tag{12}$$

g'_{ij} being the metric tensor for the new superimposed state, given by

$$g'_{ij} = \frac{\partial w_i}{\partial \Theta_j} - \Gamma^r_{ij}\, w_r + \frac{\partial w_j}{\partial \Theta_i} - \Gamma^r_{ji}\, w_r, \tag{13}$$

where w_i are the components of the small displacement vector $\mathbf{w}$ which may be expressed as

$$\mathbf{w} = w_m\, \mathbf{g}^m = w^m \mathbf{g}_m,$$

$\mathbf{g}_m$ and $\mathbf{g}^m$ being the covariant and contravariant base vectors for the deformed configuration.

The displacement components are given by

$$w_1 = u \qquad \text{and} \qquad w_2 = 0, \qquad w_3 = w. \tag{14}$$

Introduction of the derived expressions for the stresses for the finite deformation state as well as for the superimposed state, in the equations of motion for the new state, yields

$$u\left[\frac{\phi_0 \lambda_r}{r} - \frac{\lambda_\theta^2(\phi_{ac} + \phi_{as})}{r}\right] - \frac{\partial w}{\partial z}\,[\lambda_z^2(\phi_{bc} + \phi_{bs})] + \lambda_r \phi_0 \frac{\partial u}{\partial r}$$
$$+ \frac{r\lambda_z^2\, \Psi_0}{\lambda_r}\frac{\partial^2 u}{\partial z^2} = \frac{\rho r}{\lambda_r}\frac{\partial^2 u}{\partial t^2}$$

and (15)

$$\lambda_r \Psi_0 \frac{\partial^2 u}{\partial r \partial z} + \frac{\lambda_\theta^2}{r}\left(\Psi_{ac} + \Psi_{as} + \frac{\Psi_r \Psi_0}{\lambda_\theta^2}\right)\frac{\partial u}{\partial z} + \frac{\lambda_r^2 \phi_0}{r}\frac{\partial w}{\partial r}$$
$$+ \lambda_z^2 (\Psi_{bc} + \Psi_{bs}) \frac{\partial^2 w}{\partial z^2} = \frac{\rho}{\lambda_z}\frac{\partial^2 w}{\partial t^2}.$$

In deriving these equations it has been assumed that the radial stress is negligibly small compared to the circumferential and longitudinal stresses (cf. Vaishnav et al. [17]).

The displacement components u and w are given by

$$u = u_1\, e^{i\omega t} \qquad \text{and} \qquad w = w_1\, e^{i\omega t} \tag{16}$$

in which

$$u_1 = u_1(r, z) \qquad \text{and} \qquad w_1 = w_1(r, z).$$

To make the equations (15) dimensionless, let us introduce the following dimensionless quantities:

$$\bar{\phi}_{ac} = \frac{\phi_{ac}}{\phi_0}, \qquad \bar{\phi}_{as} = \frac{\phi_{as}}{\phi_0}, \qquad \bar{u}_1 = \frac{u_1}{r}, \qquad \bar{\Psi}_{bc} = \frac{\Psi_{bc}}{\phi_0},$$
$$\bar{\Psi}_{bs} = \frac{\Psi_{bs}}{\phi_0}, \qquad \bar{\omega} = \sqrt{\frac{\rho}{\phi_0}}\, r\omega, \qquad \bar{z} = \frac{z}{r}, \qquad \bar{\Psi}_0 = \frac{\Psi_0}{\phi_0}, \tag{17}$$
$$\bar{w}_1 = \frac{w_1}{r}.$$

Setting $\partial \bar{u}_1/\partial r = 0 = \partial \bar{w}_1/\partial r$ (for a thin vessel) Eqs. (15) in terms of the above written dimensionless quantities read

$$\beta_1 \bar{u}_1 - \beta_2 \frac{d\bar{w}_1}{d\bar{z}} + \beta_3 \frac{d^2 \bar{u}_1}{d\bar{z}^2} = 0$$

and

$$\beta_4 \frac{d\bar{u}_1}{d\bar{z}} + \beta_5 \frac{d^2 \bar{w}_1}{d\bar{z}^2} + \beta_6 \bar{w}_1 = 0. \tag{18}$$

The calculated expressions for the coefficients β_1, β_2, etc., are included in Appendix 2.

Boundary Conditions and the Method of Solution

Let l be the length of the blood vessel segment under consideration. We assume that one of the plane ends, $z=0$ (say) is held fixed, while the other end $z=l$ is subjected to a longitudinal force. These conditions may be mathematically formulated as

$$\bar{u}_1 = 0 = \bar{w}_1 \qquad \text{on } z=0$$

and (19)

$$\bar{u}_1 = 0,\ s'_z = s_0 \qquad \text{on } z=l,$$

in which s_0 is a constant and represents the magnitude of the force experienced by the end $z=l$ of the vessel segment.

Let us assume the solutions of (19), in the forms

$$\bar{u}_1 = Ae^{mz}, \qquad \bar{w}_1 = Be^{mz}$$

where A, B and m are undetermined constants.

The characteristic equation is

$$m^4 + K_1 m^2 + K_2 = 0 \tag{20}$$

with

$$K_1 = \frac{\beta_1\beta_5 + \beta_6\beta_3 + \beta_2\beta_4}{\beta_3\beta_5}$$

and

$$K_2 = \frac{\beta_1\beta_6}{\beta_3\beta_5}. \tag{21}$$

The equation (20) is a biquadratic equation whose roots are given by

$$m^2 = \frac{-K_1 \pm \sqrt{K_1^2 - 4K_2}}{2} = m_1, m_2 \text{ (say)}.$$

i.e.,

$$m = \pm\sqrt{m_1},\ \pm\sqrt{m_2}.$$

So the expression for $\bar{u}_1$ can be written as

$$\bar{u}_1 = A_1 e^{\sqrt{m_1}z} + A_2 e^{\ \sqrt{m_1}z} + A_3 e^{\sqrt{m_2}z} + A_4 e^{\ \sqrt{m_2}z}, \tag{22}$$

in which A_1, A_2, A_3 and A_4 are arbitrary constants. Substituting the expressions for $\bar{u}_1$ in the first equation of (18) and then integrating, we obtain

$$\bar{W}_1 = \frac{A_1}{\beta_2}\left[\frac{\beta_1}{\sqrt{m_1}} + \beta_3\sqrt{m_1}\right]e^{\sqrt{m_1}z} - \frac{A_2}{\beta_2}\left[\frac{\beta_1}{\sqrt{m_1}} + \sqrt{m_1}\,\beta_3\right]e^{-\sqrt{m_1}z}$$
$$+\frac{A_3}{\beta_2}\left[\frac{\beta_1}{\sqrt{m_2}} + \beta_3\sqrt{m_2}\right]e^{\sqrt{m_2}z} - \frac{A_4}{\beta_2}\left[\frac{\beta_1}{\sqrt{m_2}} + \beta_3\sqrt{m_2}\right]e^{-\sqrt{m_2}z}. \quad (23)$$

By employing the boundary conditions (19) with the help of the relations (10), (22) and (23), we obtain four inhomogeneous algebraic equations, in the four unknowns A_1, A_2, A_3 and A_4. Then applying Cramer's rule, we determine the solutions of these equations as

$$\frac{A_1}{s_0} = \frac{D_1}{D'}, \quad \frac{A_2}{s_0} = \frac{D_2}{D'}, \quad \frac{A_3}{s_0} = \frac{D_3}{D'} \quad \text{and} \quad \frac{A_4}{s_0} = \frac{D_4}{D'},$$

where D' is a fourth order determinant $|a_{ij}|$ whose elements are given in Appendix 3 and the determinants D_1, D_2, D_3 and D_4 are described by

$$D_1 = \begin{vmatrix} 0 & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & a_{32} & a_{33} & a_{34} \\ 1 & a_{42} & a_{43} & a_{44} \end{vmatrix}, \quad D_2 = \begin{vmatrix} a_{11} & 0 & a_{13} & a_{14} \\ a_{21} & 0 & a_{23} & a_{24} \\ a_{31} & 0 & a_{33} & a_{34} \\ a_{41} & 1 & a_{43} & a_{44} \end{vmatrix}$$

$$D_3 = \begin{vmatrix} a_{11} & a_{12} & 0 & a_{14} \\ a_{21} & a_{22} & 0 & a_{24} \\ a_{31} & a_{32} & 0 & a_{34} \\ a_{41} & a_{42} & 1 & a_{44} \end{vmatrix} \quad \text{and} \quad D_4 = \begin{vmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & 1 \end{vmatrix}.$$

With the determination of these constants, the displacement components for the problem under consideration can be completely calculated from (22), (23) and (16). Using this further, we can obtain the stress differences by employing the relations (9).

Results and Discussion

For the purpose of numerical computation, we have made use of the four-function theory (cf. Young et al. [7]) for the kernel functions (in

10^3 dynes/cm^2) given by the following expressions (for canine middle descending thoracic aorta).

$$\bar{K}^{2222}(t) = 2c_{22}(t) = 282 + 22\exp(-0.47t^{0.47}),$$

$$\bar{K}^{2233}(t) = c_{23}(t) = 192,$$

$$\bar{K}^{3322}(t) = c_{32}(t) = 270 + 16\exp(-1.67t^{0.22})$$

$$\bar{K}^{3333}(t) = 2c_{33}(t) = 267 + 28\exp(-0.51t^{0.44}).$$

Further, we have taken

$$\lambda_1 = 1.4, \qquad \lambda_2 = 1.6 \qquad \text{and} \qquad l = 3\,\text{cms}.$$

The values of the real and imaginary parts (cf. Misra, Hartung and Mahrenholtz [20 24]) of the nondimensional stress-components have been

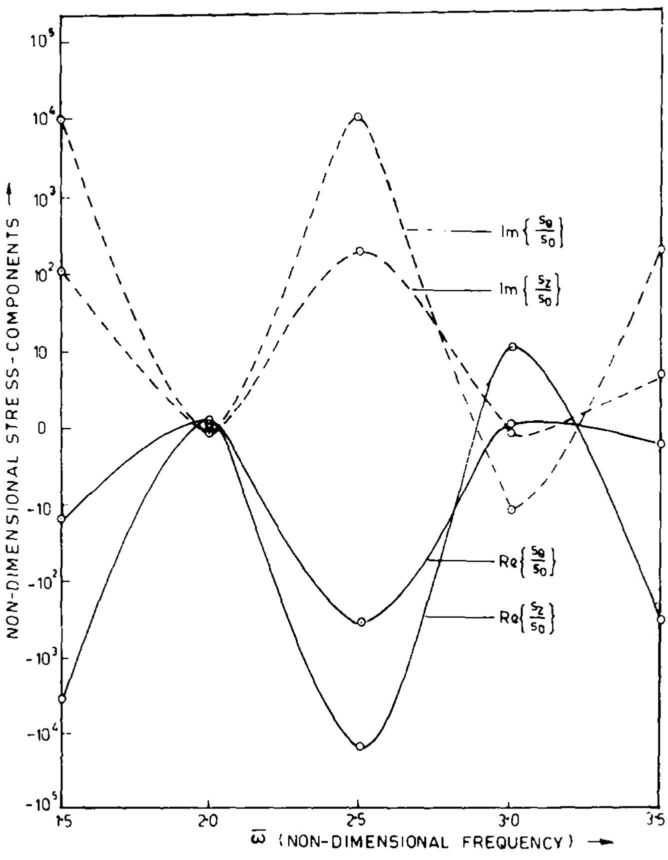

FIG. 1. Variation of the stress-components with frequency.

computed, based upon the present analysis. Variations of the nondimensional stress with nondimensional frequencies are exhibited in Fig. 1.

It is needless to mention that one can easily explore a variety of cases by employing the analysis together with the computed values because of this nondimensionalization. By assigning specific values to ρ_0, ρ, ϕ_0, etc., the actual values of the stress components can be obtained, corresponding to a specific blood vessel segment subjected to a particular longitudinal force applied at one of its ends.

Appendix 1

$$\phi_0 = (1 + 2a_0)[a_0 \bar{K}_2^2(\infty) + b_0 \bar{K}_3^2(\infty)],$$

$$\Psi_0 = (1 + 2b_0)[a_0 \bar{K}_2^3(\infty) + b_0 \bar{K}_3^3(\infty)],$$

$$\phi_{ac} = (1 + 2a_0)\, \bar{K}_2^2(0),$$

$$\phi_{as} = (1 + 2a_0)\left[\int_0^\infty \frac{d\bar{K}_2^2(s)}{ds} e^{-i\omega s}\, ds\right],$$

$$\phi_{bc} = (1 + 2a_0)\, \bar{K}_3^2(0),$$

$$\Psi_{ac} = (1 + 2b_0)\, \bar{K}_3^3(0),$$

$$\Psi_{as} = (1 + 2b_0)\left[\int_0^\infty \frac{d\bar{K}_2^3(s)}{ds} e^{-i\omega s}\, ds\right],$$

$$\Psi_{bc} = (1 + 2b_0)\, \bar{K}_3^3(0),$$

$$\Psi_{bs} = (1 + 2b_0)\left[\int_0^\infty \frac{d\bar{K}_3^3(s)}{ds} e^{-i\omega s}\, ds\right],$$

in which $\bar{K}_j^i = \bar{K}^{iijj}$ $\quad (i, j = 2, 3)$.

Appendix 2

$$\beta_1 = \left(\frac{\lambda_r}{\lambda_\theta^2} - \phi_{ac} - \phi_{as}\right)\lambda_\theta^2 + \frac{\bar{\omega}^2}{\lambda_r}, \qquad \beta_2 = (\phi_{bc} + \phi_{bs})\, \lambda_z^2,$$

$$\beta_3 = \frac{\bar{\Psi}_0 \lambda_z^2}{\lambda_r}, \qquad \beta_4 = [\bar{\Psi}_0 \lambda_r + (\bar{\Psi}_{ac} + \bar{\Psi}_{as})\, \lambda_\theta^2],$$

$$\beta_5 = \lambda_z^2(\bar{\Psi}_{bc} + \bar{\Psi}_{bs}) \qquad \text{and} \qquad \beta_6 = \frac{\bar{\omega}^2}{\lambda_z}.$$

APPENDIX 3

$$a_{11}=1=a_{12}=a_{13}=a_{14}, \qquad a_{21}=\frac{1}{\beta_2}\left(\frac{\beta_1}{\sqrt{m_1}}+\beta_3\sqrt{m_1}\right),$$

$$a_{22}=-a_{21}, \qquad a_{23}=\frac{1}{\beta_2}\left(\frac{\beta_1}{\sqrt{m_2}}+\beta_3\sqrt{m_2}\right), \qquad a_{24}=-a_{23},$$

$$a_{31}=e^{\sqrt{m_1}l}, \quad a_{32}=e^{-\sqrt{m_1}l}, \quad a_{33}=e^{\sqrt{m_2}l}, \quad a_{34}=e^{-\sqrt{m_2}l},$$

$$a_{41}=\left[(\bar{\Psi}_{ac}+\bar{\Psi}_{as})\lambda_\theta^2+\frac{\bar{\Psi}_{bc}+\bar{\Psi}_{bs}}{\beta_2\sqrt{m_1}}\lambda_z^2(\beta_1+m_1\beta_5)\right]e^{\sqrt{m_1}l},$$

$$a_{42}=\left[(\bar{\Psi}_{ac}+\bar{\Psi}_{as})\lambda_\theta^2-\frac{\bar{\Psi}_{bc}+\bar{\Psi}_{bs}}{\beta_2\sqrt{m_1}}\lambda_z^2(\beta_1+m_1\beta_5)\right]e^{-\sqrt{m_1}l},$$

$$a_{43}=\left[(\bar{\Psi}_{ac}+\bar{\Psi}_{as})\lambda_\theta^2+\frac{\bar{\Psi}_{bc}+\bar{\Psi}_{bs}}{\beta_2\sqrt{m_2}}\lambda_z^2(\beta_1+m_1\beta_5)\right]e^{\sqrt{m_2}l},$$

$$a_{44}=\left[(\bar{\Psi}_{ac}+\bar{\Psi}_{as})\lambda_\theta^2-\frac{\bar{\Psi}_{bc}+\bar{\Psi}_{bs}}{\beta_2\sqrt{m_2}}\lambda_z^2(\beta_1+m_1\beta_5)\right]e^{-\sqrt{m_2}l}.$$

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